1. [10] Given the functions $f(x) = \frac{2 - x}{x}$ and $g(x) = \frac{1}{x - 1}$:

a. [8] Find the composition $(f \circ g)(x)$ and simplify. \hspace{1cm} \textbf{Ans:} $(f \circ g)(x) = 2x - 3$

Use $f(x)$ as the input for $g(x)$ and then simplify:

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x - 1}\right) = \frac{2 - \frac{1}{x - 1}}{x - 1} = \left(\frac{2}{1} \cdot \frac{x - 1}{x - 1} - \frac{1}{x - 1}\right) \cdot \frac{x - 1}{1} = 2x - 3$$

b. [2] What is the domain of $(f \circ g)(x)$? \hspace{1cm} \textbf{Ans:} Domain of $(f \circ g)(x)$ is $(-\infty, 1) \cup (1, \infty)$

\textit{Write your answer using interval notation.}

Domain of $g(x)$: Denominator, $x - 1$, cannot be 0 so $\{x \mid x \neq 1\}$

Domain of $(f \circ g)(x)$: All real numbers

Combining these we have $\{x \mid x \neq 1\}$ In interval notation this is $(-\infty, 1) \cup (1, \infty)$

2. [14] Given the function $f(x) = \frac{1 - x}{x} + 2$:

a. [2] Find the domain of $f(x)$. \hspace{1cm} \textbf{Ans:} Domain of $f(x): (-\infty, 0) \cup (0, \infty)$

\textit{Write your answer using interval notation.}

The denominator of $f(x) = \frac{1 - x}{x} + 2$ cannot be 0. So, $x \neq 0$.

b. [8] Find the inverse $f^{-1}(x)$ \hspace{1cm} \textbf{Ans:} $f^{-1}(x) = \frac{1}{x - 1}$

Exchange the $x$ and $y$ and then simplify:

$$f(x) = \frac{1 - x}{x} + 2$$

$$y = \frac{1 - x}{x} + 2$$

\textit{To find $f^{-1}(x)$, exchange $x$ and $y$}

$$x = \frac{1 - y}{y} + 2$$

$$xy = 1 - y + 2y$$

$$xy = 1 + y$$

$$xy - y = 1$$

$$y(x - 1) = 1$$

$$y = \frac{1}{x - 1}$$
c. [4] Find the range of $f(x)$.

The range of $f(x)$ is the domain of $f^{-1}(x)$.

The denominator of $f^{-1}(x) = \frac{1}{x-1}$ cannot be 0.

So, $x \neq 1$

Notice how the function and its inverse are symmetric about the line $y = x$.

Notice that the function has a horizontal asymptote at $y = 1$. This is the only value for $y$ that is restricted.

If we write $f(x) = \frac{1-x}{x} + 2 = \frac{1-x}{x} + 2 \cdot \frac{x}{x} = \frac{x+1}{x}$

notice that $y = 1$ is the ratio of the dominant terms of the function.

Notice that $x = 0$ is a vertical asymptote of the function and so $y = 0$ is a horizontal asymptote of its inverse.

Notice that $y = 1$ is a horizontal asymptote of the function and so $x = 1$ is a vertical asymptote of its inverse.

3. [10] Solve: $4 + \log_x (8) = 2$

$\textbf{Ans: } x = 0.35$

Round your answer to 2 decimal places.

Isolate the log function and then convert to an exponential and solve.

$$4 + \log_x (8) = 2$$
$$\log_x (8) = -2$$
$$x^{-2} = 8$$
$$\frac{1}{x^2} = 8$$
$$\frac{1}{8} = x^2$$
$$\pm \sqrt{\frac{1}{8}} = x$$
$$\pm 0.353553391 = x$$

We cannot have a negative base so rounded to two decimal places this is 0.35
4. [10] Calculate \( \log \sqrt{2} \left( \frac{1}{16} \right) \).  

\[
\text{Ans: } -12
\]

This is easy to do with the change of base formula: \( \log_a b = \frac{\log_c b}{\log_c a} \)

So, \( \log \sqrt{2} \left( \frac{1}{16} \right) = \frac{\ln \left( \frac{1}{16} \right)}{\ln \left( \sqrt{2} \right)} \approx \frac{-2.772588722}{0.34657359} \approx -8 \)

If you have forgotten the change of base formula, you can solve it this way:

\[
\log \sqrt{2} \left( \frac{1}{16} \right) = y
\]

\[
(\sqrt{2})^y = \frac{1}{16}
\]

\[
\left(2^{1/2}\right)^y = \frac{1}{2^4}
\]

\[
2^{1/2}y = 2^{-4}
\]

\[
1/2y = -4
\]

\[
y = -8
\]

5. [10] Solve: \( (8^{x})^2 \cdot 32 = 2^{-x^2} \)  

\[
\text{Ans: } x = -1 \text{ or } x = -5
\]

Write the expressions so that they have the same base:

\[
(8^{x})^2 \cdot 32 = 2^{-x^2}
\]

\[
(8)^{2x} \cdot 32 = 2^{-x^2}
\]

\[
(2^{3})^{2x} \cdot 2^5 = 2^{-x^2}
\]

\[
(2)^{6x} \cdot 2^5 = 2^{-x^2}
\]

\[
(2)^{6x+5} = (2)^{-x^2}
\]

\[
6x + 5 = -x^2
\]

\[
x^2 + 6x + 5 = 0
\]

\[
(x + 5)(x + 1) = 0
\]

\[
x = -5 \text{ or } x = -1
\]
6. [10] Solve: $\log_4 (x + 4) = 1 + \log_2 (x)$.

**Round your answer to 2 decimal places.**

Use the properties of logs to combine all the log terms into a single term:

$$\log_4 (x + 4) = 1 + \log_2 (x)$$

$$\frac{\log_2 (x + 4)}{\log_2 (4)} = 1 + \log_2 (x)$$

$$\frac{\log_2 (x + 4)}{2} = 1 + \log_2 (x)$$

$$\log_2 (x + 4) = 2 + 2 \cdot \log_2 (x)$$

$$\log_2 (x + 4) - 2 \cdot \log_2 (x) = 2$$

$$\log_2 \left( \frac{x + 4}{x^2} \right) = 2$$

$$\frac{x + 4}{x^2} = 2^2$$

$$x + 4 = 4x^2$$

$$0 = 4x^2 - x - 4$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(4)(-4)}}{2(4)}$$

$$x = \frac{1 \pm \sqrt{65}}{8}$$

$$x = 1.13 \quad \text{or} \quad x = -0.88$$

$x = -0.88$ is not in the domain of the original equation (cannot take the log of a negative number) so the only solution is $x = 1.13$. 

**Ans:** $x = 1.13$